

ON CIMMINO INTEGRALS AS RESIDUES OF ZETA FUNCTIONS

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ABSTRACT. The following paper is a variation on a theme of Gianfranco Cimmino on some integral representation formulas for the solution of a linear equations system.

Cimmino was probably motivated for giving a representation formula suitable not only for theoretical investigations but also for applied computation.

In this paper we will prove that the Cimmino integrals are strictly related to the residues of some zeta-like functions associated to the linear system.

1. INTRODUCTION

Gianfranco Cimmino was born in Naples on March 12, 1908.

He received his Laurea degree in Mathematics at the University of Naples under the direction of Mauro Picone (1885-1977) in 1927.

At the end of 1939 Cimmino moved permanently to the University of Bologna to occupy the chair of Mathematical Analysis.

Cimmino died in Bologna on May 30, 1989.

Gianfranco Cimmino (1908-1989) was a student of Renato Caccioppoli (1904-1959) and Mauro Picone together with Giuseppe Scorza Dragoni (1907-1996) and Carlo Miranda (1912-1982).

Mauro Picone founded the “Istituto per le Applicazioni del Calcolo” (IAC) in 1927. Indeed, long before the introduction of digital computers, Picone had the intuition of the potential impact on real-life problems of the combination of computational methods with mathematical abstraction.

Probably the influence of Picone ideas and Cesari papers [2] and [3] on numerical solution of linear systems leads Cimmino to be interested to the numerical treatment of the solutions of linear systems of algebraic equations.

Among his researches he also obtained an interesting representation of the solution of a linear system of real linear equations that now we describe.

Let

$$(1) \quad Ax = b$$

be a linear system of n equation and n unknown, where the unknown values x_1, \dots, x_n are the components if the column vector $x \in \mathbb{R}^n$ and A is non singular matrix with real coefficient of order n .

The well known Cramer's rule say that if $D = \det(A) \neq 0$ then

$$(2) \quad x_i = \frac{D_i}{D}, \quad i = 1, \dots, n,$$

where D_i is the determinant of the matrix obtained replacing the i -th column of the matrix A with the column vector b .

The alternative representation of (2) given by Cimmino is

$$(3) \quad x_i = \frac{C_i}{C}, \quad i = 1, \dots, n,$$

where

$$(4) \quad C = \int_{S^{n-1}} \|A^t u\|^{-n} du,$$

and

$$(5) \quad C_i = n \int_{S^{n-1}} \|A^t u\|^{-n-2} \langle b, u \rangle \langle A^t x, e_i \rangle du.$$

The integration here is made with respect to the standard $(n - 1)$ -dimensional measure on S^{n-1} , the boundary of the Euclidean unit ball in \mathbb{R}^n , $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ stand respectively for the Euclidean norm and the Euclidean inner product on \mathbb{R}^n and e_1, \dots, e_n is the canonical basis of \mathbb{R}^n , and A^t denote the transpose of the matrix A .

In [4], [5] and [7] Cimmino gives a probabilistic argument which justify the existence of such kind of formulas and in [6] he also give an elementary but not trivial proof of (3). See also ([8]) for some applications. We refer to [1] for a detailed discussion and a background information on Cimmino's papers in the field of the numerical analysis.

The purpose of this paper is to show that the Cimmino ideas fit nicely into the theory of the residues of zeta-like functions, a powerful tool coming from the analytic number theory.

Namely we will show that (4) and (5) are the integral representation of the residues of suitable zeta-like function associate to the matrix A and the vector b of the linear system (1); see Theorem 7.2 for the complete statement.

It should be noted that formulas 4 and 5 actually are due to Jacobi: cf. [10].

2. THETA FUNCTIONS AND THEIR MELLIN TRANSFORM

Let us begin with the following (almost trivial) observation:

Proposition 2.1. *Let f be a continuous complex function defined on the real interval $[0, 1]$. Assume that for some constants $R, \alpha, \beta \in \mathbb{R}$*

with $\alpha > \beta$ we have

$$f(t) = Rt^{-\alpha} + O(t^{-\beta}), \quad t \rightarrow 0^+.$$

Then, given $s \in \mathbb{C}$, the integral

$$g(s) = \int_0^1 f(t)t^s \frac{dt}{t}$$

converges absolutely on the half space $\operatorname{Re}(s) > \alpha$ and extends to a meromorphic function on the half space $\operatorname{Re}(s) > \beta$ having a simple pole at $s = \alpha$ with residue R .

Proof. Since $f(t) = O(t^{-\alpha})$ then $g(s)$ converges and is holomorphic when $\operatorname{Re}(s) > \alpha$. If we denote

$$h(t) = f(t) - Rt^{-\alpha}$$

then

$$g(s) = \int_0^1 f(t)t^s \frac{dt}{t} = \frac{R}{s - \alpha} + \int_0^1 h(t)t^s \frac{dt}{t}.$$

Since $h(t) = O(t^{-\beta})$ the last integral define a holomorphic function on the half space $\operatorname{Re}(s) > \beta$ and we are done.

□

Thus the singularities of the analytic function $g(s)$ describe the behaviour of the function $f(t)$ as $t \rightarrow 0^+$.

Let us recall that $C(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ denotes respectively the space of the continuous complex function on \mathbb{R}^n and the space of the absolutely integrable complex functions with respect to the Lebesgue measure on \mathbb{R}^n .

Given $f \in L^1(\mathbb{R}^n)$ the *Fourier transform* of f is the function \hat{f} defined by the formula

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \langle x, y \rangle} dx.$$

Observe that

$$\hat{f}(0) = \int_{\mathbb{R}^n} f(x) dx.$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of smooth functions rapidly decreasing at the infinity together with their derivatives of all orders. Given $\alpha, \beta > 0$ two positive real number we denote by

$$\mathcal{S}_{n+\alpha}^{n+\beta}(\mathbb{R}^n)$$

the space of all measurable function such that

$$\sup_{x \in \mathbb{R}^n} |f(x)| (1 + \|x\|^{n+\alpha}) < +\infty$$

and

$$\sup_{y \in \mathbb{R}^n} |\hat{f}(y)| (1 + \|y\|^{n+\beta}) < +\infty.$$

Of course $\mathcal{S}_{n+\alpha}^{n+\beta}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is the intersection of all the spaces $\mathcal{S}_{n+\alpha}^{n+\beta}(\mathbb{R}^n)$ when α and β varies on all the positive real numbers. Since

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{\alpha, \beta > 0} \mathcal{S}_{n+\alpha}^{n+\beta}(\mathbb{R}^n)$$

is useful to set

$$\mathcal{S}_{n+\infty}^{n+\infty}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n).$$

Let $\alpha, \beta > 0$, finite or infinite, be fixed .

Let $f \in \mathcal{S}_{n+\alpha}^{n+\beta}(\mathbb{R}^n)$ be an arbitrary function.

By the very elementary approach to the (Riemann) integration theory we have

$$\hat{f}(0) = \int_{\mathbb{R}^n} f(x) dx = \lim_{t \rightarrow 0^+} t^n \sum_{\omega \in \mathbb{Z}^n} f(t\omega).$$

It is then natural to define for $t > 0$ and $d > 0$

$$\begin{aligned} \theta_d(f, t) &= \sum_{\omega \in \mathbb{Z}^n} f(t^{1/d}\omega), \\ \theta_d^*(f, t) &= \sum'_{\omega \in \mathbb{Z}^n} f(t^{1/d}\omega) = \theta_d(f, t) - f(0), \end{aligned}$$

(where $\sum'_{\omega \in \mathbb{Z}^n}$, as usual, stands for $\sum_{\omega \in \mathbb{Z}^n \setminus \{0\}}$) and considering the *Mellin transform*.

$$\xi_d(f, s) = \int_0^{+\infty} \theta_d^*(f, t) t^s \frac{dt}{t}$$

(The introduction of the constant d will simplify some computations).

Our hope is to study the Riemann approximation (6) looking at the residues of $\xi_d(f, s)$.

Indeed we have

Theorem 2.1. *Let $\alpha, \beta > 0$ be given and let $f \in \mathcal{S}_{n+\alpha}^{n+\beta}(\mathbb{R}^n)$ be an arbitrary function. Then the integral defining $\xi_d(f, s)$ converges in the strip $n/d < \operatorname{Re}(s) < (n+\alpha)/d$ and the function $\xi_d(f, s)$ extends to a meromorphic function on the strip $-\beta/d < \operatorname{Re}(s) < (n+\alpha)/d$. having exactly two simple poles respectively at $s = n/d$ with residue $\hat{f}(0)$ and at $s = 0$ with residue $-f(0)$. Moreover we have the functional equation*

$$\xi_d(f, s) = \xi_d\left(\hat{f}, n/d - s\right), \quad -\frac{\beta}{d} < \operatorname{Re}(s) < \frac{n+\alpha}{d}.$$

The *proof* of the theorem above follows closely the lines of (some of) the standard proof of the functional equation for the classical zeta functions used in number theory, but the *statement* of the theorem in the form above is not so common. We refer [11] for a detailed proof.

Observe that $\theta_d^*(f, t) = \theta_1^*(f, t^{1/d})$ and hence, by a simple change of integration variable, $\xi_d(f, s) = d\xi_1(f, ds)$. It follows that the function $\xi_d(f, s)$ has a pole at $s = n/d$ if, and only if, the function $\xi_1(f, s)$ has a pole at $s = n$ with the same residue.

3. ZETA FUNCTIONS MACHINERY

The following proposition describe the hearth of our approach to the treatment of Cimmino integrals.

Proposition 3.1. *Let $\alpha, \beta, d > 0$ be given and let $f \in \mathcal{S}(\mathbb{R}^n)$ be an arbitrary function. Assume that in the strip $-\beta/d < \operatorname{Re}(s) < (n+\alpha)/d$ the function $\xi_d(f, s)$ decomposes as*

$$\xi_d(f, s) = G_f(s)Z_f(s)$$

with $Z_f(s)$ being meromorphic with exactly a single simple pole at $s = n/d$.

Then

$$\begin{aligned} Z_f(0) \operatorname{Res}_{s=0} G_f(s) &= -f(0), \\ G_f(n/d) \operatorname{Res}_{s=n/d} Z_f(s) &= \hat{f}(0). \end{aligned}$$

Proof. By the previous theorem $\xi_d(f, s)$ has a simple pole at $s = 0$ and

$$-f(0) = \operatorname{Res}_{s=0} \xi_d(f, s) = \operatorname{Res}_{s=0} G_f(s)Z_f(s).$$

Since $Z_f(s)$ is holomorphic at $s = 0$ then

$$-f(0) = Z_f(0) \operatorname{Res}_{s=0} G_f(s).$$

Replacing f with \hat{f} , we have

$$-\hat{f}(0) = \operatorname{Res}_{s=0} \xi_d(\hat{f}, s).$$

Using the functional equation $\xi_d(f, s) = \xi_d(\hat{f}, n/d - s)$ we obtain

$$\hat{f}(0) = -\operatorname{Res}_{s=0} \xi_d(\hat{f}, s) = -\operatorname{Res}_{s=0} \xi_d(f, n/d - s) = \operatorname{Res}_{s=n/d} \xi_d(f, s).$$

Since $\xi_d(f, s)$ and $Z_f(s)$ have a simple pole at $s = n/d$ then necessarily $G_f(s)$ is holomorphic at $s = n/d$ and hence

$$\hat{f}(0) = G_f(n/d) \operatorname{Res}_{s=n/d} Z_f(s),$$

as desired.

□

In the sequel we will apply the above proposition to functions f which gives a decompositions $\xi_d(f, s) = G_f(s)Z_f(s)$ where the function $G_f(s)$ is an algebraic combination of elementary functions and functions of

the form $\Gamma(as + b)$, being $\Gamma(z)$ the Euler gamma function, and $Z_f(s)$ is representable when $\operatorname{Re}(s) \gg 0$ as (generalized) Dirichlet series

$$Z_f(s) = \sum_{k=1}^{\infty} c_k \lambda_k^{-s},$$

where $c_k, \lambda_k \in \mathbb{R}$, $\lambda_1 < \lambda_2 < \dots$, and $\lambda_k \rightarrow +\infty$ as $k \rightarrow +\infty$.

4. QUADRATIC FORMS

We begin considering the well known classical example of the Gaussian integrals associated to quadratic form.

Let us recall that $\Gamma(s)$ denotes the Euler gamma function; it is a meromorphic function on \mathbb{C} which for $\operatorname{Re}(s) > 0$ satisfies

$$\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt.$$

We assume the reader knows all the properties of such a function.

Let Q be a real symmetric matrix of order n . We set

$$q_Q(x) = \langle Qx, x \rangle$$

Assume that Q is positive definite and consider the function

$$g_Q(x) = e^{-\pi q_Q(x)}$$

then $g_Q \in \mathcal{S}(\mathbb{R}^n)$ and a standard argument yields

$$\hat{g}_Q(y) = \frac{1}{\sqrt{\det Q}} g_{Q^{-1}}(y).$$

In particular

$$\hat{g}_Q(0) = \frac{1}{\sqrt{\det Q}}.$$

Since $q_Q(tx) = t^2 q_Q(x)$ then, choosing $d = 2$,

$$\theta_2^*(g_Q, t) = \sum'_{\omega \in \mathbb{Z}^n} e^{-t\pi q_Q(\omega)}$$

and hence, when $\operatorname{Re}(s) > n/2$,

$$\xi_2(g_Q, s) = \sum'_{\omega \in \mathbb{Z}^n} \int_0^{+\infty} e^{-t\pi q_Q(\omega)} t^s \frac{dt}{t}.$$

The change of variable $u = t\pi q_Q(\omega)$ yields

$$\begin{aligned} \xi_2(g_Q, s) &= \pi^{-s} \left(\int_0^{+\infty} e^{-u} u^s \frac{du}{u} \right) \sum'_{\omega \in \mathbb{Z}^n} q_Q(\omega)^{-s} \\ &= \pi^{-s} \Gamma(s) \sum'_{\omega \in \mathbb{Z}^n} q_Q(\omega)^{-s}. \end{aligned}$$

The function defined for $\text{Re}(s) > n/2$ by

$$\zeta(q_Q, s) = \sum'_{\omega \in \mathbf{Z}^n} q_Q(\omega)^{-s}$$

is the *Epstein zeta function* associated to the quadratic form q_Q .

Since $g_Q(x) \in \mathcal{S}(\mathbb{R}^n)$ then, by Theorem 2.1, $\xi_2(g_Q, s)$ is a meromorphic function on \mathbb{C} having exactly two simple poles at $s = n/2$ and $s = 0$ with residues respectively $R = 1$ and $R = -1$.

Since the function $\pi^{-s}\Gamma(s)$ never vanishes on \mathbb{C} and the function $\Gamma(s)$ has a simple pole at $s = 0$ with residue $R = 1$ it follows that $\zeta(q_Q, s)$ extends to a meromorphic function on \mathbb{C} having exactly a simple pole at $s = n/2$.

By Proposition 3.1 and the relation

$$\Gamma(s+1) = s\Gamma(s)$$

we obtain that the residue of $\zeta(q_Q, s)$ at $s = n/2$ is

$$\text{Res}_{s=n/2} \zeta(q_Q, s) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{\sqrt{\det Q}} = \frac{n}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{\sqrt{\det Q}}$$

and that

$$\zeta(q_Q, s) = -1.$$

All that is well known to number theorists.

But now consider two quadratic forms, q_Q and q_S with Q and S symmetric matrices with Q positive definite and set

$$g_{Q,S}(x) = q_S(x)e^{-\pi q_Q(x)}.$$

Then $g_{Q,S} \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 4.1. *The Fourier transform of $g_{Q,S}(x)$ is*

$$\begin{aligned} \hat{g}_{Q,S}(y) &= -\frac{1}{\sqrt{\det Q}} g_{Q^{-1}, Q^{-1}SQ^{-1}}(y) + \frac{\text{Tr}(Q^{-1}S)}{2\pi\sqrt{\det Q}} g_{Q^{-1}}(y) \\ &= -\frac{1}{\sqrt{\det Q}} q_{Q^{-1}SQ^{-1}}(y) e^{-\pi q_{Q^{-1}}(y)} + \frac{\text{Tr}(Q^{-1}S)}{2\pi\sqrt{\det Q}} e^{-\pi q_{Q^{-1}}(y)}, \end{aligned}$$

where $\text{Tr}(Q^{-1}S)$ denotes the trace of the matrix $Q^{-1}S$.

In particular we have

$$\int_{\mathbf{R}^n} g_{Q,S}(x) dx = \hat{g}_{Q,S}(0) = \frac{\text{Tr}(Q^{-1}S)}{2\pi\sqrt{\det Q}}.$$

Proof. The function

$$e^{-\pi\|x\|^2},$$

where

$$\|x\|^2 = \langle x, x \rangle = \sum_{i=1}^n x_i^2,$$

coincide with its Fourier transform. Set

$$h_{ij}(x) = x_i x_j e^{-\pi\|x\|^2}.$$

Then, the standard properties of the Fourier transform yields

$$\hat{h}_{ij}(y) = -\frac{1}{4\pi^2} D_{y_i} D_{y_j} e^{-\pi\|y\|^2},$$

where D_{y_i} denote the operator of derivation with respect to the variable y_i .

We compute

$$\begin{aligned} h_{ij}(y) &= y_i y_j e^{-\pi\|y\|^2}, \\ D_{y_j} h_{ij}(y) &= -2\pi y_j e^{-\pi\|y\|^2}, \\ D_{y_i} D_{y_j} h_{ij}(y) &= -2\pi D_{y_i} (y_j e^{-\pi\|y\|^2}) = 4\pi^2 e^{-\pi\|y\|^2} - 2\pi \delta_{ij} e^{-\pi\|y\|^2}, \\ \hat{h}_{ij}(y) &= -\frac{1}{4\pi^2} D_{y_i} D_{y_j} e^{-\pi\|y\|^2} = -e^{-\pi\|y\|^2} + \frac{1}{2\pi} \delta_{ij} e^{-\pi\|y\|^2}. \end{aligned}$$

Let denote by I_n the identity matrix of order n . Let $C = (c_{ij})$ be a real symmetric matrix of order n and set

$$h_C(x) = g_{I_n, C}(x) = q_C(x) e^{-\pi\|x\|^2} = \sum_{i,j=1}^n c_{ij} h_{ij}(x)$$

Then

$$\begin{aligned} \hat{h}_C(y) &= \sum_{i,j=1}^n c_{ij} \hat{h}_{ij}(y) \\ &= \sum_{i,j=1}^n -c_{ij} e^{-\pi\|y\|^2} + \sum_{i,j=1}^n \frac{1}{2\pi} c_{ij} \delta_{ij} e^{-\pi\|y\|^2} \\ &= -q_C(y) e^{-\pi\|y\|^2} + \frac{1}{2\pi} \text{Tr}(C) e^{-\pi\|y\|^2} \end{aligned}$$

Let $Q^{\frac{1}{2}}$ be the unique positive definite symmetric matrix such that

$$\left(Q^{\frac{1}{2}}\right)^2 = Q$$

and let $Q^{-\frac{1}{2}}$ be its inverse.

Set

$$f(x) = h_C(x)$$

where $C = Q^{-\frac{1}{2}}SQ^{-\frac{1}{2}}$. We have

$$\text{Tr}(C) = \text{Tr}(Q^{-\frac{1}{2}}SQ^{-\frac{1}{2}}) = \text{Tr}(Q^{-\frac{1}{2}}Q^{-\frac{1}{2}}S) = \text{Tr}(Q^{-1}S)$$

and hence

$$\hat{f}(y) = -q_C(y)e^{-\pi\|y\|^2} + \frac{1}{2\pi} \text{Tr}(Q^{-1}S)e^{-\pi\|y\|^2}.$$

We also have

$$\begin{aligned} q_C(Q^{\frac{1}{2}}x) &= q_S(x), \\ q_C(Q^{-\frac{1}{2}}y) &= q_{Q^{-1}SQ^{-1}}(y), \\ \|Q^{\frac{1}{2}}y\| &= q_Q(y). \end{aligned}$$

It follows that

$$g_{Q,S}(x) = f(Q^{\frac{1}{2}}x)$$

and hence

$$\begin{aligned} \hat{g}_{Q,S}(y) &= \frac{1}{\sqrt{\det Q}} \hat{f}(Q^{-\frac{1}{2}}y) \\ &= -\frac{1}{\sqrt{\det Q}} q_{Q^{-1}SQ^{-1}}(y)e^{-\pi q_{Q^{-1}}(y)} + \frac{\text{Tr}(Q^{-1}S)}{2\pi\sqrt{\det Q}} e^{-\pi q_{Q^{-1}}(y)}, \end{aligned}$$

as desired.

□

Let now compute $\theta_2^*(g_{Q,S}, t)$ and $\xi_2(g_{Q,S}, s)$.

Since $q_Q(tx) = t^2q_Q(x)$ and $q_S(tx) = t^2q_S(x)$ we have

$$\theta_2^*(g_{Q,S}, t) = \sum'_{\omega \in \mathbf{Z}^n} tq_S(\omega)e^{-t\pi q_S(\omega)},$$

and

$$\begin{aligned} \xi_2(g_{Q,S}, s) &= \int_0^{+\infty} \sum'_{\omega \in \mathbf{Z}^n} tq_S(\omega)e^{-t\pi q_S(\omega)} t^s \frac{dt}{t} \\ &= \sum'_{\omega \in \mathbf{Z}^n} \int_0^{+\infty} q_S(\omega)e^{-t\pi q_S(\omega)} t^{s+1} \frac{dt}{t}. \end{aligned}$$

By the change of variable $u = t\pi q_S(\omega)$ we obtain

$$\begin{aligned} \xi_2(g_{Q,S}, s) &= \pi^{-(s+1)} \left(\int_0^{+\infty} e^{-u} u^{s+1} \frac{dt}{t} \right) \sum'_{\omega \in \mathbf{Z}^n} q_S(\omega) q_Q(\omega)^{-(s+1)} \\ &= \pi^{-(s+1)} \Gamma(s+1) \sum'_{\omega \in \mathbf{Z}^n} q_S(\omega) q_Q(\omega)^{-(s+1)}. \end{aligned}$$

Set

$$\zeta(q_Q, q_S, s) = \sum'_{\omega \in \mathbf{Z}^n} q_S(\omega) q_Q(\omega)^{-s}$$

The meromorphic function $\pi^{-(s+1)}\Gamma(s+1)$ never vanishes on \mathbb{C} . Since $g_{Q,S}(0) = 0$ by Theorem 2.1 the function $\xi_2(g_{Q,S}, s)$ is a meromorphic function on \mathbb{C} having exactly a simple pole at $s = n/2$ and hence the sum defining $\zeta(q_Q, q_S, s)$ converges for $\text{Re}(s) > n/2$ and extends to a meromorphic function on \mathbb{C} having exactly a simple pole at $s = n/2$. Then proposition 3.1 and proposition 4.1 give:

Theorem 4.1. *Let Q and S be two symmetric matrices of order n with Q positive definite. Then the sum defining the zeta function $\zeta(q_Q, q_S, s)$ converges absolutely for $\text{Re}(s) > n/2 + 1$.*

The zeta function $\zeta(q_Q, q_S, s)$ extends to a meromorphic function on \mathbb{C} having exactly a simple pole at $s = n/2 + 1$ with residue

$$\text{Res}_{s=n/2+1} \zeta(q_Q, q_S, s) = \frac{1}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{\text{Tr}(Q^{-1}S)}{\sqrt{\det Q}}.$$

From the equality

$$\hat{g}_{Q,S}(y) = -\frac{1}{\sqrt{\det Q}} g_{Q^{-1}, Q^{-1}SQ^{-1}}(y) + \frac{\text{Tr}(Q^{-1}S)}{2\pi\sqrt{\det Q}} g_{Q^{-1}}(y)$$

proved in Proposition 4.1 we obtain

$$\begin{aligned} \xi_2(\hat{g}_{Q,S}, s) &= -\frac{1}{\sqrt{\det Q}} \pi^{-(s+1)} \Gamma(s+1) \zeta(q_{Q^{-1}}, q_{Q^{-1}SQ^{-1}}, s+1) \\ &\quad + \frac{\text{Tr}(Q^{-1}S)}{2\sqrt{\det Q}} \pi^{-(s+1)} \Gamma(s) \zeta(q_{Q^{-1}}, s). \end{aligned}$$

Inserting such expressions in the functional equation

$$\xi_2\left(g_{Q,S}, \frac{n}{2} - s\right) = \xi_2(\hat{g}_{Q,S}, s)$$

and dividing by π we obtain:

Theorem 4.2. *Let Q and S be as in theorem 4.1. The zeta function $\zeta(q_Q, q_S, s)$ satisfies the functional equation*

$$\begin{aligned} &\pi^{-(\frac{n}{2}-s)} \Gamma\left(\frac{n}{2} + 1 - s\right) \zeta\left(q_Q, q_S, \frac{n}{2} + 1 - s\right) \\ &\quad + \frac{1}{\sqrt{\det Q}} \pi^{-s} \Gamma(s+1) \zeta(q_{Q^{-1}}, q_{Q^{-1}SQ^{-1}}, s+1) \\ &= \frac{\text{Tr}(Q^{-1}S)}{2\sqrt{\det Q}} \pi^{-s} \Gamma(s) \zeta(q_{Q^{-1}}, s). \end{aligned}$$

5. LATTICES

A *lattice* in \mathbb{R}^n is a set of the form

$$L = \{A\omega \mid \omega \in \mathbf{Z}^n\}$$

where $A \in GL(n, \mathbb{R})$ is a real invertible matrix of order n .

If A_1 is an other matrix such that $A_1(\mathbf{Z}^n) = L$ then $U = AA_1^{-1}(\mathbf{Z}^n) \subset \mathbf{Z}^n$. It follows that the invertible matrix U has integral coefficients and

hence $|\det U| = 1$, that is $|\det A| = |\det A_1|$. We define the volume of the lattice L as

$$|L| = |\det A|.$$

By the argument given above the definition does not depend on the choice of the matrix A .

If $L = A(\mathbf{Z}^n)$ is a lattice the *dual lattice* is the lattice L^\wedge associate to the inverse of the transpose of the matrix A . For convenience we also set

$$\hat{A} = (A^t)^{-1}$$

Given a lattice $L \subset \mathbb{R}^n$ and a positive definite symmetric matrix Q of order n we define

$$\zeta_L(q_Q, s) = \sum'_{\omega \in L} q_Q(\omega)^{-s}$$

and if S is any symmetric matrix we also define

$$\zeta_L(q_Q, q_S, s) = \sum'_{\omega \in L} q_S(\omega) q_Q(\omega)^{-s}.$$

Of course we have

$$\zeta_{\mathbf{Z}^n}(q_Q, s) = \zeta(q_Q, s)$$

and

$$\zeta_{\mathbf{Z}^n}(q_Q, q_S, s) = \zeta(q_Q, q_S, s).$$

If $L = A(\mathbf{Z}^n)$ with $A \in GL(n, \mathbb{R})$ then we have

$$\zeta_L(q_Q, s) = \zeta(q_{A^t Q A}, s),$$

where A^t denotes the transpose of the matrix A , and

$$\zeta_L(q_Q, q_S, s) = \zeta(q_{A^t Q A}, q_{A^t S A}, s).$$

We also have

$$\sqrt{\det(A^t Q A)} = |\det A| \sqrt{\det Q} = |L| \sqrt{\det A}$$

and

$$\begin{aligned} \text{Tr}((A^t Q A)^{-1} (A^t S A) (A^t Q A)^{-1}) &= \text{Tr}(A^{-1} (Q^{-1} S Q^{-1}) A) \\ &= \text{Tr}(Q^{-1} S Q^{-1}). \end{aligned}$$

Applying the theta-zeta machinery to the function

$$g_{A,Q,S}(x) = g_{Q,S}(Ax) = q_S(Ax) e^{-\pi q_Q(Ax)},$$

observing that the Fourier transform of $g_{A,Q,S}(x)$ is

$$\hat{g}_{A,Q,S}(y) = \frac{1}{|\det A|} \hat{g}_{Q,S}(\hat{A}y),$$

it follows that the results of the previous section generalize:

Theorem 5.1. *Let $L \subset \mathbb{R}^n$ be a lattice and Let Q and S be two symmetric matrices of order n with Q positive definite. Then the sum defining the zeta functions $\zeta_L(q_Q, s)$ and $\zeta_L(q_Q, q_S, s)$ converges absolutely respectively for $\operatorname{Re}(s) > n/2$ and $\operatorname{Re}(s) > n/2 + 1$.*

The zeta functions $\zeta_L(q_Q, s)$ and $\zeta_L(q_Q, q_S, s)$ extends to a meromorphic function on \mathbb{C} having exactly a simple pole respectively at $s = n/2$ and $s = n/2 + 1$ with residues

$$\operatorname{Res}_{s=n/2} \zeta_L(q_Q, s) = \frac{n}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{|L| \sqrt{\det Q}},$$

and

$$\operatorname{Res}_{s=n/2+1} \zeta_L(q_Q, q_S, s) = \frac{1}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{\operatorname{Tr}(Q^{-1}S)}{|L| \sqrt{\det Q}}.$$

Theorem 5.2. *Let $L \subset \mathbb{R}^n$, Q and S be as in theorem 5.1. The zeta functions $\zeta_L(q_Q, s)$ and $\zeta_L(q_Q, q_S, s)$ satisfy the functional equations respectively*

$$\begin{aligned} & \pi^{-(\frac{n}{2}-s)} \Gamma\left(\frac{n}{2} - s\right) \zeta_L\left(q_Q, \frac{n}{2} - s\right) \\ &= \frac{1}{|L| \sqrt{\det Q}} \pi^{-s} \Gamma(s) \zeta_{L^\wedge}(q_{Q^{-1}}, s). \end{aligned}$$

and

$$\begin{aligned} & \pi^{-(\frac{n}{2}-s)} \Gamma\left(\frac{n}{2} + 1 - s\right) \zeta_L\left(q_Q, q_S, \frac{n}{2} + 1 - s\right) \\ &+ \frac{1}{|L| \sqrt{\det Q}} \pi^{-s} \Gamma(s+1) \zeta_{L^\wedge}(q_{Q^{-1}}, q_{Q^{-1}SQ^{-1}}, s+1) \\ &= \frac{\operatorname{Tr}(Q^{-1}S)}{2|L| \sqrt{\det Q}} \pi^{-s} \Gamma(s) \zeta_{L^\wedge}(q_{Q^{-1}}, s). \end{aligned}$$

6. INTEGRAL REPRESENTATION

Let q_Q and q_S be two quadratic form with Q and S symmetric matrices and Q positive definite.

We now will give such residues as integrals over the boundary of the unit ball in \mathbb{R}^n .

We already observed that

$$\int_{\mathbb{R}^n} e^{-\pi q_Q(x)} dx = \frac{1}{\sqrt{\det Q}}$$

and by Proposition 4.1 we also have

$$\int_{\mathbb{R}^n} q_S(x) e^{-\pi q_Q(x)} dx = \frac{\operatorname{Tr}(Q^{-1}S)}{2\pi \sqrt{\det Q}}.$$

If $A \in GL(n, \mathbb{R})$ then a simple change of variable gives

$$\int_{\mathbb{R}^n} e^{-\pi q_Q(Ax)} dx = \frac{1}{|\det A| \sqrt{\det Q}}$$

and

$$\int_{\mathbb{R}^n} q_S(Ax) e^{-\pi q_Q(Ax)} dx = \frac{\text{Tr}(Q^{-1}S)}{2\pi |\det A| \sqrt{\det Q}}.$$

Let us recall that given $f \in L^1(\mathbb{R}^n)$ the integration by polar coordinates gives

$$\int_{\mathbb{R}^n} f(x) dx = \int_{S^{n-1}} \left(\int_0^{+\infty} f(ru) r^{n-1} dr \right) du,$$

where du is the Euclidean (hyper-)surface measure on the unit sphere S^{n-1} .

Using such formula we obtain

$$\begin{aligned} \frac{\text{Tr}(Q^{-1}S)}{2\pi |\det A| \sqrt{\det Q}} &= \int_{\mathbb{R}^n} q_S(Ax) e^{-\pi q_Q(Ax)} dx \\ &= \int_{S^{n-1}} \left(\int_0^{+\infty} q_S(rAu) e^{-\pi q_Q(rAu)} r^{n-1} dr \right) du \\ &= \int_{S^{n-1}} q_S(Au) \left(\int_0^{+\infty} e^{-\pi q_Q(Au)} r^{n+2} \frac{dr}{r} \right) du \end{aligned}$$

By the substitution $\pi r^2 q_Q(Au) = t$ in the inner integral we obtain

$$\begin{aligned} &\int_{S^{n-1}} q_S(Au) \left(\int_0^{+\infty} e^{-\pi q_Q(Au)} r^{n+2} \frac{dr}{r} \right) du \\ &= \frac{1}{2} \int_{S^{n-1}} q_S(Au) \left(\int_0^{+\infty} e^{-t} \left(\frac{t}{\pi q_Q(Au)} \right)^{n/2+1} \frac{dt}{t} \right) du \\ &= \frac{1}{2} \pi^{-(n/2+1)} \Gamma\left(\frac{n}{2} + 1\right) \int_{S^{n-1}} q_S(Au) q_Q(Au)^{-(n/2+1)} du \end{aligned}$$

and hence

$$\int_{S^{n-1}} q_S(Au) q_Q(Au)^{-(n/2+1)} du = \left(\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \right) \frac{\text{Tr}(Q^{-1}S)}{|\det A| \sqrt{\det Q}}.$$

When $S = Q$ we obtain

$$\int_{S^{n-1}} q_Q(Au)^{-n/2} du = n \left(\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \right) \frac{1}{|\det A| \sqrt{\det Q}}.$$

Replacing s with $s/2$ in the formulas of the residues in Theorem 5.1 we obtain

$$\text{Res}_{s=n} \zeta_L(q_Q, s/2) = n \left(\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \right) \frac{1}{|L| \sqrt{\det Q}},$$

and

$$\operatorname{Res}_{s=n+2} \zeta_L(q_Q, q_S, s/2) = \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{\operatorname{Tr}(Q^{-1}S)}{|L| \sqrt{\det Q}}.$$

Thus we obtained:

Theorem 6.1. *Let q_Q and q_S be two quadratic form with Q and S symmetric matrices and Q positive definite. Let $A \in GL(n, \mathbb{R})$ and set $L = A(\mathbf{Z}^n)$. Then we have*

$$\operatorname{Res}_{s=n} \zeta_L(q_Q, s/2) = \int_{S^{n-1}} q_Q(Au)^{-n/2} du.$$

and

$$\operatorname{Res}_{s=n+2} \zeta_L(q_Q, q_S, s/2) = \int_{S^{n-1}} q_S(Au) q_Q(Au)^{-(n/2+1)} du,$$

7. LINEAR SYSTEMS

We recall that \hat{A} denotes the inverse of the transpose of the matrix A .

Given $A \in GL(\mathbb{R}^n)$ we define

$$\zeta(A, s) = \sum'_{\omega \in \mathbf{Z}^n} \|A\omega\|^{-2s}$$

and given also $b \in \mathbb{R}^n$ we define the vector value zeta function

$$\zeta(A, b, s) = \sum'_{\omega \in \mathbf{Z}^n} \|A\omega\|^{-2s} \langle b, \omega \rangle A\omega.$$

Of course we have

$$\zeta(A, s) = \zeta_L(q_{I_n}, s)$$

with $L = A(\mathbf{Z}^n)$, I_n the identity matrix of order n and hence, by Theorem 5.1, we have

$$\operatorname{Res}_{s=n/2} \zeta(A, s) = \frac{n}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{|\det A|}$$

But we also have:

Theorem 7.1. *The series defining $\zeta(A, b, s)$ converges for $\operatorname{Re}(s) > n/2 + 1$ and the function $\zeta(A, b, s)$ extends to a (vector value) meromorphic function on \mathbb{C} having only a simple pole at $s = n/2 + 1$ with residue*

$$\operatorname{Res}_{s=n/2+1} \zeta(A, b, s) = \frac{1}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{|\det A|} \hat{A}b.$$

Moreover, if $c \in \mathbb{R}^n$ then we have the functional equation

$$\begin{aligned} & \pi^{-(\frac{n}{2}-s)} \Gamma\left(\frac{n}{2} + 1 - s\right) \left\langle \zeta\left(A, b, \frac{n}{2} + 1 - s\right), Ac \right\rangle \\ & + \frac{\pi^{-s}}{|\det A|} \Gamma(s+1) \left\langle \hat{A}b, \zeta\left(\hat{A}, c, s+1\right) \right\rangle \\ & = \frac{\langle b, c \rangle}{2|\det A|} \pi^{-s} \Gamma(s) \zeta\left(\hat{A}, s\right). \end{aligned}$$

Proof. Given $u, v \in \mathbb{R}^n$, with $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ we denote by $u \otimes v$ the symmetric matrix (w_{ij}) of order n with entries given by $w_{ij} = u_i v_j$. Then we have

$$\text{Tr}(u \otimes v) = \langle u, v \rangle.$$

Now observe that

$$\langle \zeta(A, b, s), Ac \rangle = \zeta_L(q_{I_n}, q_S, s)$$

where $L = A(\mathbf{Z}^n)$, I_n is the identity matrix of order n and

$$S = \hat{A}b \otimes Ac.$$

We have

$$\text{Tr}(\hat{A}b \otimes Ac) = \langle \hat{A}b, Ac \rangle = \langle b, c \rangle$$

and hence, the formulas for the residue and the functional equation of the function $\langle \zeta(A, b, \frac{n}{2} + 1 - s), Ac \rangle$ follows from Theorem 5.1 and Theorem 5.2

□

We are now ready to state and prove the reformulation of Cimmino's results.

Theorem 7.2. *Let*

$$Ax = b$$

be a linear system of n equation and n unknown, where the unknown values x_1, \dots, x_n are the components if the column vector $x \in \mathbb{R}^n$ and A is non singular matrix with real coefficient of order n .

Then we have

$$x_i = \frac{R_i}{R}, \quad i = 1, \dots, n,$$

where

$$R = \text{Res}_{s=n} \zeta(A^t, s/2)$$

and

$$R_i = n \text{Res}_{s=n+2} \langle \zeta(A^t, b, s/2), e_i \rangle, \quad i = 1, \dots, n,$$

where e_1, \dots, e_n is the canonical basis of \mathbb{R}^n .

Moreover we have the identities

$$R = \int_{S^{n-1}} \|A^t u\|^{-n} du,$$

and

$$R_i = n \int_{S^{n-1}} \|A^t u\|^{-n-2} \langle b, u \rangle \langle A^t x, e_i \rangle du.$$

Proof. By assumption

$$x = A^{-1}b$$

and we have

$$x_i = \langle x, e_i \rangle, \quad i = 1, \dots, n$$

where e_1, \dots, e_n is the canonical basis of \mathbb{R}^n . As previously observed we have

$$R = n \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{|\det A^t|} = n \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{|\det A|}$$

and for $i = 1, \dots, n$ we have

$$\begin{aligned} R_i &= n \frac{1}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{|\det A^t|} \langle \hat{A}^t b, e_i \rangle \\ &= \frac{n}{2} \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \right) \frac{1}{|\det A|} \langle A^{-1}b, e_i \rangle \end{aligned}$$

and hence

$$\frac{R_i}{R} = \langle A^{-1}b, e_i \rangle = x_i,$$

as required.

The last assertions of the Theorem follows from Theorem 6.1.

□

Our tour around Cimmino's ideas is completed.

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